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Some Fundamental Properties of Successive Convex Relaxation Methods on LCP and Related Problems

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Abstract. General successive convex relaxation methods (SRCMs) can be used to compute the convex hull of any compact set, in an Euclidean space, described by a system of quadratic inequalities and a compact convex set. Linear complementarity problems (LCPs) make an interesting and rich class of structured nonconvex optimization problems. In this paper, we study a few of the specialized lift-and-project methods and some of the possible ways of applying the general SCRMs to LCPs and related problems.

Key words: Nonconvex quadratic optimization, Linear complementarity problem, Semidefinite programming, Global optimization, SDP relaxation, Convex relaxation, Lift-and-project procedures.

1. Introduction, SCRMs and background

Since 1960s, complementarity problems attracted a very significant attention in the theory as well as applications of operations research. See, for instance, the book on LCP [4]. In this paper, we consider various complementarity problems in the context of Successive convex relaxation methods (SCRMs) proposed by the authors [6, 7]. Since these methods can be used to compute the convex hull of any compact subset of an Euclidean space described by a system of quadratic inequalities and a compact convex set, they can be used to attack many complementarity problems from several angles.

In the special case of 0-1 optimization problems over convex sets, or more specially polytopes, there are many SCRMs based on lift-and-project techniques. We also discuss some of the relationships of general SCRMs and these more specialized algorithms in solving LCPs.

Let ℓ be an integer such that $1 < 2\ell \leqslant m$, $\boldsymbol{d} \in R^m$, and let A be a compact convex subset of R^m. Consider the convex optimization problem with complementarity conditions:

$$\left.\begin{array}{ll}\text{maximize} & \boldsymbol{d}^T\boldsymbol{u} \\ \text{subject to} & \boldsymbol{u} \in A,\ 0 \leqslant u_i,\ 0 \leqslant u_{i+\ell},\ u_i u_{i+\ell} = 0,\ \forall i \in \{1, 2, \dots, \ell\}.\end{array}\right\} \tag{1}$$

First of all, it is clear that LCP, with a known upper bound on a solution of it, is a special case of (1) (we can take $m = 2\ell$ and A as an affine subspace intersected with a large enough ball). Secondly, it is very elementary to formulate this problem as a mixed 0-1 optimization problem with convex constraints:

$$\left.\begin{array}{ll}\text{maximize} & \boldsymbol{c}^T\boldsymbol{v} \\ \text{subject to} & \boldsymbol{v} \in C_0,\ v_i \in \{0,\ 1\},\ \forall i \in \{m+1, m+2, \dots, n\},\end{array}\right\} \tag{2}$$

where

$$C_0 \equiv \left\{ \boldsymbol{v} = \begin{pmatrix} \boldsymbol{u} \\ v_{m+1} \\ \vdots \\ v_n \end{pmatrix} \in R^{m+\ell} : \begin{array}{l} \boldsymbol{u} \in A, \\ 0 \leqslant u_i \leqslant r_i v_{m+i}, \\ 0 \leqslant u_{i+\ell} \leqslant r_{i+\ell}(1 - v_{m+i}), \\ \forall i \in \{1, 2, \dots, \ell\} \end{array} \right\},$$

$$\boldsymbol{c} \equiv \begin{pmatrix} \boldsymbol{d} \\ \boldsymbol{0} \end{pmatrix} \in R^{m+\ell},$$

$$n \equiv m + \ell,\ r_i \geqslant \max\{u_i : \boldsymbol{u} \in A\}, i \in \{1, 2, \dots, 2\ell\}.$$

In general, we allow C_0 to be an arbitrary compact convex set in R^n. There are various successive convex relaxation methods that can be applied to such a problem.

We can represent the feasible region $F \subset R^n$ of (2) as

$$F = \{\boldsymbol{v} \in C_0 : p(\boldsymbol{v}) \leqslant 0,\ \forall p(\cdot) \in \mathcal{P}_F\},$$

where $\mathcal{P}_F$ denotes a set consisting of quadratic functions

$$(v_i^2 - v_i),\ (-v_i^2 + v_i),\ i \in \{m+1, m+2, \dots, n\}$$

on R^n.

SCRMs take as input, a compact convex subset C_0 of R^n and a set $\mathcal{P}_F$ of quadratic functions which induce a description of a compact subset F of R^n such that

$$F = \{\boldsymbol{x} \in C_0 : qf(\boldsymbol{x}; \gamma, \boldsymbol{q}, \boldsymbol{Q}) \leqslant 0,\ qf(\cdot; \gamma, \boldsymbol{q}, \boldsymbol{Q}) \in \mathcal{P}_F\}.$$

Here we denote by $qf(\cdot; \gamma, \boldsymbol{q}, \boldsymbol{Q})$, the quadratic function $(\gamma + 2\boldsymbol{q}^T\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x})$. Note that the variable $\boldsymbol{x}$ is irrelevant outside a context and it will always be clear what the variable vector is, from the context. For every compact convex relaxation

$C \subseteq C_0$ of F and every subset D of $\overline{D} \equiv \{\boldsymbol{d} \in R^n : \|\boldsymbol{d}\| = 1\}$,

$$\mathcal{P}^2(C, D) \equiv \{-(\boldsymbol{d}^T\boldsymbol{v} - \alpha(C_0, \boldsymbol{d}))(\bar{\boldsymbol{d}}^T\boldsymbol{v} - \alpha(C, \bar{\boldsymbol{d}})) : \boldsymbol{d} \in D,\ \bar{\boldsymbol{d}} \in \overline{D}\},$$

$$\widehat{\mathcal{N}}(C, D) \equiv \left\{ \boldsymbol{v} \in C_0 : \begin{array}{l} \exists \boldsymbol{V} \in \mathcal{S}^n \text{ such that} \\ \gamma + 2\boldsymbol{q}^T\boldsymbol{v} + \boldsymbol{Q} \bullet \boldsymbol{V} \leqslant 0, \\ \forall qf(\cdot; \gamma, \boldsymbol{q}, \boldsymbol{Q}) \in \mathcal{P}_F \cup \mathcal{P}^2(C, D) \end{array} \right\}$$

(a Semi-Infinite LP relaxation of F),

$$\widehat{\mathcal{N}}_+(C, D) \equiv \left\{ \boldsymbol{v} \in C_0 : \begin{array}{l} \exists \boldsymbol{V} \in \mathcal{S}^n \text{ such that } \begin{pmatrix} 1 & \boldsymbol{v}^T \\ \boldsymbol{v} & \boldsymbol{V} \end{pmatrix} \in \mathcal{S}^{1+n}_+, \\ \gamma + 2\boldsymbol{q}^T\boldsymbol{v} + \boldsymbol{Q} \bullet \boldsymbol{V} \leqslant 0, \\ \forall qf(\cdot; \gamma, \boldsymbol{q}, \boldsymbol{Q}) \in \mathcal{P}_F \cup \mathcal{P}^2(C, D) \end{array} \right\}$$

(a Semi-Infinite SDP relaxation of F),

where $\alpha(C, \boldsymbol{d}) \equiv \max\{\boldsymbol{d}^T\boldsymbol{v} : \boldsymbol{v} \in C\}$ for every $\boldsymbol{d} \in \overline{D}$. Let $\mathcal{S}^n$ and $\mathcal{S}^{1+n}_+$ denote the set of $n \times n$ symmetric matrices and the set of $(1+n) \times (1+n)$ symmetric positive semidefinite matrices, respectively. The corresponding variants of Successive Semi-Infinite LP Relaxation Method (SSILPRM) and Successive SDP Relaxation Method (SSDPRM) can be written as follows.

ALGORITHM 1.1. (SSILPRM)
Step 0: Choose a $D_0 \subseteq \overline{D}$. Let $k = 0$.
Step 1: If $C_k =$ (the convex hull of F), then stop.
Step 2: Let $C_{k+1} = \widehat{\mathcal{N}}(C_k, D_0)$.
Step 3: Let $k = k + 1$, and go to Step 1.

ALGORITHM 1.2. (SSDPRM)
Steps 0, 1 and 3: The same as the Steps 0, 1 and 3 of Algorithm 1.1.
Step 2: Let $C_{k+1} = \widehat{\mathcal{N}}_+(C_k, D_0)$.

To connect these algorithms to the Lovász-Schrijver procedures, we need to introduce some additional notation. Let S be a subset of R^n. Then $\text{conv}(S)$ denotes the convex hull of S and $\text{cone}(S)$ denotes the convex cone generated by all nonnegative, linear combinations of the elements of S. If S is a convex cone in R^n then its dual is defined as

$$S^* \ \equiv\ \{\boldsymbol{s} \in R^n :\ \langle \boldsymbol{x}, \boldsymbol{s} \rangle\ \geqslant\ 0,\ \forall\, \boldsymbol{x} \in S\}.$$

For every pair of closed convex cones $\mathcal{K}$ and $\mathcal{T}$ in R^{1+n}, define

$$\mathcal{M}(\mathcal{K}, \mathcal{T}) \equiv \left\{ \boldsymbol{Y} = \begin{pmatrix} \lambda & \lambda\boldsymbol{v}^T \\ \lambda\boldsymbol{v} & \lambda\boldsymbol{V} \end{pmatrix} : \begin{array}{l} \lambda \geqslant 0,\ \boldsymbol{v} \in C_0,\ \boldsymbol{V} \in \mathcal{S}^n, \\ v_i = V_{ii},\ i \in \{m+1, m+2, \ldots, n\}, \\ \boldsymbol{v}^T\boldsymbol{Y}\boldsymbol{w} \geqslant 0,\ \forall \boldsymbol{v} \in \mathcal{T}^*,\ \forall \boldsymbol{w} \in \mathcal{K}^* \end{array} \right\},$$

$$\mathcal{M}_+(\mathcal{K}, \mathcal{T}) \equiv \left\{ \boldsymbol{Y} = \begin{pmatrix} \lambda & \lambda\boldsymbol{v}^T \\ \lambda\boldsymbol{v} & \lambda\boldsymbol{V} \end{pmatrix} : \begin{array}{l} \lambda \geqslant 0,\ \boldsymbol{v} \in C_0,\ \boldsymbol{V} \in \mathcal{S}^n, \boldsymbol{Y} \in \mathcal{S}^{1+n}_+ \\ v_i = V_{ii},\ i \in \{m+1, m+2, \ldots, n\}, \\ \boldsymbol{v}^T\boldsymbol{Y}\boldsymbol{w} \geqslant 0,\ \forall \boldsymbol{v} \in \mathcal{T}^*,\ \forall \boldsymbol{w} \in \mathcal{K}^* \end{array} \right\}.$$

Let $\mathcal{T}_0$ and $\mathcal{K}_0$ be closed convex cones given by

$$\mathcal{T}_0^* = \text{cone}\left(\left\{\begin{pmatrix} \alpha(C_0, \boldsymbol{d}) \\ -\boldsymbol{d} \end{pmatrix} \in R^{1+n} : \boldsymbol{d} \in D_0\right\}\right),$$
$$\mathcal{K}_0 = \left\{\begin{pmatrix} \lambda \\ \lambda \boldsymbol{v} \end{pmatrix} \in R^{1+n} : \boldsymbol{v} \in C_0,\ \lambda \geqslant 0\right\}.$$

(Note that $\mathcal{T}_0$ itself is defined as the dual of $\mathcal{T}_0^*$.) If $C \subseteq C_0$ is a compact convex relaxation of F and

$$\mathcal{K} = \left\{\begin{pmatrix} \lambda \\ \lambda \boldsymbol{v} \end{pmatrix} \in R^{1+m} : \boldsymbol{v} \in C,\ \lambda \geqslant 0\right\},$$

then

$$\widehat{\mathcal{N}}(C, D_0) = \left\{\boldsymbol{v} \in R^n : \begin{pmatrix} 1 & \boldsymbol{v}^T \\ \boldsymbol{v} & \boldsymbol{V} \end{pmatrix} \in \mathcal{M}(\mathcal{K}, \mathcal{T}_0)\right\},$$
$$\widehat{\mathcal{N}}_+(C, D_0) = \left\{\boldsymbol{v} \in R^n : \begin{pmatrix} 1 & \boldsymbol{v}^T \\ \boldsymbol{v} & \boldsymbol{V} \end{pmatrix} \in \mathcal{M}_+(\mathcal{K}, \mathcal{T}_0)\right\}.$$

Algorithms 1.1 and 1.2 specialized to (2) with $\mathcal{P}_F = \{v_i^2 - v_i,\ -v_i^2 + v_i,\ i \in \{m+1, m+2, \ldots, n\}\}$ can be stated in the following forms, which are essentially the Lovász-Schrijver procedures.

ALGORITHM 1.1H (Homogeneous form of Algorithm 1.1)
Step 1: Choose a $D_0 \subseteq \overline{D}$. Define $\mathcal{T}_0$ and $\mathcal{K}_0$ as above. Let $k = 0$.
Step 2: If $\mathcal{K}_k = \text{cone}\left(\left\{\begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} : \boldsymbol{v} \in F\right\}\right)$ then stop.
Step 3: Let $\mathcal{K}_{k+1} = \{\boldsymbol{Y}\boldsymbol{e}_0 : \boldsymbol{Y} \in \mathcal{M}(\mathcal{K}_k, \mathcal{T}_0)\}$.
Step 4: Let $k = k + 1$, and go to Step 1.

ALGORITHM 1.2H (Homogeneous form of Algorithm 1.2)
Steps 0, 1 and 3: The same as Steps 0, 1 and 3 of Algorithm 1.1H, respectively.
Step 2: Let $\mathcal{K}_{k+1} = \{\boldsymbol{Y}\boldsymbol{e}_0 : \boldsymbol{Y} \in \mathcal{M}_+(\mathcal{K}_k, \mathcal{T}_0)\}$.

In this paper $\boldsymbol{e}_j$ denotes the jth unit vector and $\boldsymbol{e}$ denotes the vector of all ones (the dimensions of the vectors will be clear from the context). Let $\boldsymbol{M} \in R^{\ell \times \ell}$, $\boldsymbol{q} \in R^{\ell}$ be given. Consider the LCP in the following form.

$$\text{(LCP) Find } \boldsymbol{x}, \boldsymbol{s} \text{ such that } \begin{aligned} \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q} &= \boldsymbol{s}, \\ \boldsymbol{x} \geqslant \boldsymbol{0}&,\ \boldsymbol{s} \geqslant \boldsymbol{0}, \\ x_i s_i &= 0,\ \forall i \in \{1, 2, \ldots, \ell\}. \end{aligned}$$

Suppose we are given $\mathcal{B}(\boldsymbol{\xi}, r) \equiv \left\{\boldsymbol{u} \in R^{2\ell} : \|\boldsymbol{u} - \boldsymbol{\xi}\| \leqslant r\right\}$, an Euclidean ball containing a solution of the LCP. More specifically, we assume that if LCP has a solution (or solutions) then it has at least one solution inside this ball. (In the case of rational data $(\boldsymbol{M}, \boldsymbol{q})$, we can take $\mathcal{B}$ centered at the origin with the radius bounded above by a polynomial function of the "bit size" of the data $(\boldsymbol{M}, \boldsymbol{q})$.) For the rest of this section, we assume that the Euclidean ball with center $\boldsymbol{\xi} \equiv \boldsymbol{0}$ and the radius r (r is assumed given) contains some solution of the LCP.

Under the boundedness assumption above, it is particularly easy to model any LCP as a 0-1 mixed integer programming problem, since the only nonlinear constraints of LCP can be expressed as

$$x_i = 0, \quad \text{or} \quad s_i = 0, \ \forall i \in \{1, 2, \dots, \ell\}.$$

Balas' method [1] can be directly applied to such formulations. We can also apply some variants of the Sherali-Adams reformulation linearization technique (RLT) [12] or the Lovász-Schrijver procedures [9] to the mixed integer programming feasibility problem:

$$\begin{aligned}\text{Find } \boldsymbol{x}, \boldsymbol{s} \text{ and } \boldsymbol{z} \text{ such that } \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q} &= \boldsymbol{s}, \\ \boldsymbol{0} \leqslant \boldsymbol{x} &\leqslant r\boldsymbol{z}, \ \boldsymbol{0} \leqslant \boldsymbol{s} \leqslant r(\boldsymbol{e} - \boldsymbol{z}), \\ \boldsymbol{z} &\in \{0, 1\}^{\ell}.\end{aligned}$$

Note that we can eliminate the variable vector $\boldsymbol{s}$ from the formulation and apply the SSILPR and SSDPR Methods to the following formulation:

$$\begin{aligned}&\boldsymbol{0} \leqslant \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q} \leqslant r(\boldsymbol{e} - \boldsymbol{z}), \\ &\boldsymbol{0} \leqslant \boldsymbol{z} \leqslant \boldsymbol{e}, \ \boldsymbol{0} \leqslant \boldsymbol{x} \leqslant r\boldsymbol{z}, \\ &z_i^2 - z_i \leqslant 0, \quad -z_i^2 + z_i \leqslant 0, \quad i \in \{1, 2, \dots, \ell\}.\end{aligned}$$

To apply the SCRMs, we can take

$$\begin{aligned}C_0 &\equiv \left\{ \boldsymbol{v} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{pmatrix} \in R^n : \begin{array}{l} \boldsymbol{0} \leqslant \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q} \leqslant r(\boldsymbol{e} - \boldsymbol{z}), \\ \boldsymbol{0} \leqslant \boldsymbol{z} \leqslant \boldsymbol{e}, \ \boldsymbol{0} \leqslant \boldsymbol{x} \leqslant r\boldsymbol{z} \end{array} \right\}, \\ &m \equiv \ell, \ n \equiv 2\ell, \\ \mathcal{P}_F &\equiv \left\{ (v_i^2 - v_i), \ (-v_i^2 + v_i), \ i \in \{m+1, m+2, \dots, n\} \right\}.\end{aligned}$$

Both algorithms, SSILPRM and SSDPRM presented above, terminate in at most ℓ steps. This fact can be proved easily, using the results of Balas [1], Sherali and Adams [12], Lovász and Schrijver [9], or [6, 7]. For computational experience on similar algorithms for similar problems see [3, 13, 15, 16] and [17]. For a comparison and short review of various related convex relaxations for 0-1 mixed integer programming problems see [5].

Next, we give the details of a proof of such a convergence result when the methods are applied to a formulation of Pardalos and Rosen [11]. They homogenize the vector $\boldsymbol{q}$ with a new continuous variable α, then they maximize α.

$$\left.\begin{aligned}(\text{MIP}_\alpha) \quad &\text{maximize } \alpha \\ &\text{subject to } \boldsymbol{0} \leqslant \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}\alpha \leqslant \boldsymbol{e} - \boldsymbol{z}, \\ &\qquad \boldsymbol{0} \leqslant \boldsymbol{x} \leqslant \boldsymbol{z}, \ 0 \leqslant \alpha \leqslant 1, \ \boldsymbol{z} \in \{0, 1\}^{\ell}.\end{aligned}\right\}$$

Note that

$$\begin{pmatrix} \bar{\alpha} \\ \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{z}} \end{pmatrix} \equiv \boldsymbol{0}$$

is feasible in (MIP$_\alpha$) and, it is easy to see that (MIP$_\alpha$) has an optimal solution with $\alpha^* > 0$ iff the (LCP) has a solution (or solutions) [11]. Moreover, if

$$\begin{pmatrix} \alpha^* \\ \boldsymbol{x}^* \\ \boldsymbol{z}^* \end{pmatrix}$$

is an optimal solution of (MIP$_\alpha$) with $\alpha^* > 0$ then $\boldsymbol{x}^*/\alpha^*$ solves the (LCP) [11]. One advantage of (MIP$_\alpha$) is that it does not require the introduction of large, data dependent constants (such as r_i or r in some of the previous approaches we mentioned) or their a priori estimates. Now, we take

$$C_0 \equiv \left\{ \boldsymbol{v} = \begin{pmatrix} \alpha \\ \boldsymbol{x} \\ \boldsymbol{z} \end{pmatrix} \in R^{1+2\ell} : \begin{array}{l} \boldsymbol{0} \leqslant \boldsymbol{Mx} + \boldsymbol{q}\alpha \leqslant \boldsymbol{e} - \boldsymbol{z}, \\ \boldsymbol{0} \leqslant \boldsymbol{x} \leqslant \boldsymbol{z},\ 0 \leqslant \alpha \leqslant 1 \end{array} \right\},$$

$$m \equiv \ell + 1,\ n \equiv 2\ell + 1,$$

$$\mathcal{P}_F \equiv \left\{ (v_i^2 - v_i),\ (-v_i^2 + v_i),\ i \in \{m+1, m+2, \dots, n\} \right\}.$$

We have an analog of a very elementary but also a key lemma (Lemma 1.3 of [9]) of Lovász and Schrijver (and their proof technique is adapted here). In what follows, we refer to the vectors in the space of $\mathcal{K}_k$ by $\boldsymbol{v}$. At the same time, we refer to different subvectors of $\boldsymbol{v}$ by different names, such as $\boldsymbol{x}$, α etc., to keep the correspondence of elements of $\boldsymbol{v}$ and the original formulation of F clearer. The proof of Lemma 1.3 of [9] leads to the following analogous result in our case.

LEMMA 1.3. *Let $D_0 \supseteq \{\pm\boldsymbol{e}_{m+1}, \pm\boldsymbol{e}_{m+2}, \dots, \pm\boldsymbol{e}_n\}$. Then the sequence of convex cones $\{\mathcal{K}_k : k \geqslant 0\}$ given by Algorithm 1.1H satisfies*

$$\mathcal{K}_{k+1} \subseteq (\mathcal{K}_k \cap \{\boldsymbol{v} : x_i = 0\}) + (\mathcal{K}_k \cap \{\boldsymbol{v} : (\boldsymbol{Mx} + \boldsymbol{q}\alpha)_i = 0\}),$$

for every $i \in \{1, 2, \dots, \ell\}$, and for every $k \geqslant 0$.

Proof. Let

$$\boldsymbol{w} \equiv \begin{pmatrix} 1 \\ \bar{\alpha} \\ \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{z}} \end{pmatrix} \in \mathcal{K}_{k+1}.$$

Fix $j \in \{1, 2, \dots, \ell\}$ arbitrarily. By the definition of D_0 and $\mathcal{T}_0$, the unit vector $\boldsymbol{e}_0$ is in $\mathcal{T}_0$. Hence, by the definition of $\mathcal{M}(\mathcal{K}_k, \mathcal{T}_0)$, $\mathcal{K}_{k+1} \subseteq \mathcal{K}_k$ for every $k \geqslant 0$.

Therefore, $\boldsymbol{w} \in \mathcal{K}_k$. If $\bar{x}_j = 0$ or $(\boldsymbol{Mx} + \boldsymbol{q}\alpha)_j = 0$ then the statement of the lemma clearly holds. So, without loss of generality, we assume $\bar{x}_j > 0$ and $(\boldsymbol{Mx}+\boldsymbol{q}\alpha)_j > 0$. Let $\boldsymbol{Y} \in \mathcal{M}(\mathcal{K}_k, \mathcal{T}_0)$ such that $\boldsymbol{w} = \boldsymbol{Ye}_0$. By our choice of the cone $\mathcal{T}_0$, we conclude that $\boldsymbol{Ye}_{n+j}$ and $\boldsymbol{Y}(\boldsymbol{e}_0 - \boldsymbol{e}_{n+j})$ are both in $\mathcal{K}_k$. Note that

$$\boldsymbol{w} = \hat{\boldsymbol{w}} + \tilde{\boldsymbol{w}},$$

where $\hat{\boldsymbol{w}} \equiv \boldsymbol{Ye}_{n+j}$ and $\tilde{\boldsymbol{w}} \equiv \boldsymbol{Y}(\boldsymbol{e}_0 - \boldsymbol{e}_{n+j})$. We will refer to the $\boldsymbol{x}$ and z parts of the vector $\hat{\boldsymbol{w}}$ by $\hat{\boldsymbol{x}}$, $\hat{z}$ etc. (Similarly for $\tilde{\boldsymbol{w}}$.) First, since by the definition of $\mathcal{M}(\mathcal{K}_k, \mathcal{T}_0)$, $v_i = V_{ii}$ for every $i \in \{m+1, m+2, \ldots, n\}$, we have $\tilde{z}_j = 0$ which implies $\tilde{x}_j = 0$. Therefore, $\tilde{\boldsymbol{w}}$ lies in the cone $\left(\mathcal{K}_0 \cap \left\{\boldsymbol{v} : x_j = 0\right\}\right)$. Second, since $\bar{x}_j > 0$, $\bar{z}_j$ must be positive. Therefore, $(1/\bar{z}_j)\hat{\boldsymbol{w}} \in \mathcal{K}_0$. Since $v_i = V_{ii}$ for every $i \in \{m+1, m+2, \ldots, n\}$, $\hat{z}_j = \bar{z}_j$. So,

$$\frac{1}{\bar{z}_j}\begin{pmatrix}\hat{\alpha}\\ \hat{\boldsymbol{x}}\\ \hat{z}\end{pmatrix} \in C_k,$$

with its z_j entry equal to 1. Thus, $(\boldsymbol{M}\hat{\boldsymbol{x}} + \boldsymbol{q}\hat{\alpha})_j = 0$. Hence, $\hat{\boldsymbol{w}}$ is in the cone $\left(\mathcal{K}_k \cap \left\{\boldsymbol{v} : (\boldsymbol{Mx} + \boldsymbol{q}\alpha)_j = 0\right\}\right)$. Since the argument above is independent of the index j the proof is complete. □

Note that the conclusion of the above lemma also applies to the SSDPR Method since SSDPR Method yields at least as tight relaxations as the SSILPR Method. Also note that the inclusion in Lemma 1.3 can sometimes be strict for the SSILPR and SSDPR Methods. Below, F denotes the set of feasible solutions of (MIP$_\alpha$).

THEOREM 1.4. *Both algorithms, Algorithm 1.1H and 1.2H terminate in ℓ iterations when applied to the formulation (MIP$_\alpha$) with our choice of $\mathcal{P}_F$, C_0 and D_0 above.*

Proof. First note that

$$\mathrm{conv}(F) \subseteq \left\{\begin{pmatrix}\alpha\\ \boldsymbol{x}\\ z\end{pmatrix} \in R^n : \begin{pmatrix}1\\ \alpha\\ \boldsymbol{x}\\ z\end{pmatrix} \in \mathcal{K}_k\right\}, \quad \forall k \geqslant 0.$$

Next, let $i, j \in \{1, 2, \ldots, \ell\}$, $i \neq j$. Since $\boldsymbol{x} \geqslant \boldsymbol{0}$ and $\boldsymbol{Mx} + \boldsymbol{q}\alpha \geqslant \boldsymbol{0}$, for all $\boldsymbol{v} \in \mathcal{K}_k$, for every $k \geqslant 0$,

$$\begin{aligned}&\left[(\mathcal{K}_k \cap \{\boldsymbol{v} : x_i = 0\}) + (\mathcal{K}_k \cap \{\boldsymbol{v} : (\boldsymbol{Mx} + \boldsymbol{q}\alpha)_i = 0\})\right] \cap \left\{\boldsymbol{v} : x_j = 0\right\}\\ &\quad = \left(\mathcal{K}_k \cap \left\{\boldsymbol{v} : x_i = 0,\ x_j = 0\right\}\right) + \left(\mathcal{K}_k \cap \left\{\boldsymbol{v} : x_j = 0,\ (\boldsymbol{Mx} + \boldsymbol{q}\alpha)_i = 0\right\}\right).\end{aligned}$$

Similarly, for the intersection with $\left\{\boldsymbol{v} : (\boldsymbol{Mx} + \boldsymbol{q}\alpha)_j = 0\right\}$. Now, we apply Lemma 1.3 repeatedly to conclude that $\mathcal{K}_\ell$ is the homogenization of the convex hull of all solutions of the LCP that lie in the original relaxation C_0. □

2. SCRMs applied to a smaller formulation of LCP with explicit treatment of the disjunctive constraints

Now, we modify Pardalos-Rosen formulation and consider a formulation with fewer variables and constraints.

$$\begin{array}{lll}(\mathrm{LCP}_\alpha) & \text{maximize} & \alpha \\ & \text{subject to} & \boldsymbol{Mx}+\boldsymbol{q}\alpha \geqslant \boldsymbol{0},\ \boldsymbol{x} \geqslant \boldsymbol{0},\ \alpha \geqslant 0, \\ & & \boldsymbol{e}^T(\boldsymbol{M}+\boldsymbol{I})\boldsymbol{x}+(\boldsymbol{e}^T\boldsymbol{q}+1)\alpha \leqslant 1, \\ & & x_i(\boldsymbol{Mx}+\boldsymbol{q}\alpha)_i = 0,\ i \in \{1,2,\ldots,\ell\}.\end{array}$$

It is easy to see that

$$\begin{pmatrix}\bar{\boldsymbol{x}}\\ \bar{\alpha}\end{pmatrix} \equiv \boldsymbol{0}$$

is feasible in (LCP_α), and it is also easy to observe that (LCP_α) has an optimal solution with $\alpha^* > 0$ iff the (LCP) has a solution(or solutions). Moreover, if

$$\begin{pmatrix}\boldsymbol{x}^*\\ \alpha^*\end{pmatrix}$$

is an optimal solution of (LCP_α) with $\alpha^* > 0$ then $\boldsymbol{x}^*/\alpha^*$ solves the (LCP).

Sherali, Krishnamurty and Al-Khayyal [13] also address solving LCPs via lift-and-project methods. Their main concern seems to be practical efficiency for general LCPs (without assuming a very special structure of the data $(\boldsymbol{M}, \boldsymbol{q})$). To this end, in addition to the Sherali-Adams RLT, they employ subgradient methods, branch-and-bound algorithms and implicit enumeration techniques. Their application of RLT to their 0-1 mixed integer programming formulation also leads to a convex hull representation in at most ℓ steps.

Sherali, Krishnamurty and Al-Khayyal's mixed integer programming formulation does not use upperbounds like r_i or r. However, their results either rely on the assumptions like "the set $\{\boldsymbol{x} \in R^\ell : \boldsymbol{Mx}+\boldsymbol{q} \geqslant \boldsymbol{0}, \boldsymbol{x} \geqslant \boldsymbol{0}\}$ is bounded" (Proposition 2.1 of [13]) or the existence of a constant U large enough to allow adding a constraint like $\boldsymbol{e}^T\boldsymbol{x} \leqslant U$ (Remark 2.2 of [13]). At least from a theoretical point of view our approach in this section is stronger, in that our approach does not require similar assumptions. Nevertheless, we cannot make any claims about even the mild computational superiority of our approach.

We explicitly include the variable vector $\boldsymbol{s}$ in our discussion in this section, for the sake of presentation. Let

$$C_0 \equiv \left\{ \boldsymbol{v} = \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{s}\\ \alpha\end{pmatrix} \in R^{2\ell+1} : \begin{array}{l}\boldsymbol{s} = \boldsymbol{Mx}+\boldsymbol{q}\alpha \geqslant \boldsymbol{0},\ \boldsymbol{x} \geqslant \boldsymbol{0},\ \alpha \geqslant 0, \\ \boldsymbol{e}^T(\boldsymbol{M}+\boldsymbol{I})\boldsymbol{x}+(\boldsymbol{e}^T\boldsymbol{q}+1)\alpha \leqslant 1,\end{array} \right\}.$$

In this section, we will describe another successive convex relaxation method based on the ideas of Balas [1], Lovász and Schrijver [9]. This method will use only

Linear Programming (LP) relaxations. We describe the method in the original space of F and C_0. Let $\mathcal{F}(C_0)$ denote the set of facet defining inequalities for C_0. $\mathcal{F}(C_0)$ is the input of the algorithm which we introduce now.

ALGORITHM 2.1.
Step 0: $k \equiv 0$.
Step 1: $\mathcal{F}(C_{k+1}) \equiv \mathcal{F}(C_k)$.

Step 2: For every inequality

$$-\sum_{i=1}^{\ell} (u_i x_i + u_{\ell+i} s_i) - u_{2\ell+1}\alpha \leqslant u_0$$

in $\mathcal{F}(C_k)$ and every $j \in \{1, 2, \dots, \ell\}$ solve the LP problems

$$(P_j) \quad \text{minimize} \quad \boldsymbol{u}^T \boldsymbol{\xi}^{(j)}$$
$$\text{subject to} \quad \xi_j^{(j)} = 1,\ \xi_{\ell+j}^{(j)} = 0,\ \boldsymbol{\xi}^{(j)} \in \mathcal{K}_k,$$

and

$$(P_{\ell+j}) \quad \text{minimize} \quad \boldsymbol{u}^T \boldsymbol{\xi}^{(\ell+j)}$$
$$\text{subject to} \quad \xi_j^{(\ell+j)} = 0,\ \xi_{\ell+j}^{(\ell+j)} = 1,\ \boldsymbol{\xi}^{(\ell+j)} \in \mathcal{K}_k,$$

where $\mathcal{K}_k$ is the homogenization of C_k as described in Section 1. If (P_j) is infeasible then add the equation $x_j = 0$ (or the inequality $x_j \leqslant 0$, since the inequality $x_j \geqslant 0$ is already included) to $\mathcal{F}(C_{k+1})$. If $(P_{\ell+j})$ is infeasible then add the equation $s_j = 0$ to $\mathcal{F}(C_{k+1})$. Otherwise, let $(\boldsymbol{\xi}^{(j)})^*$ and $(\boldsymbol{\xi}^{(\ell+j)})^*$ denote the optimal solutions of (P_j) and $(P_{\ell+j})$ respectively. Define $y_j \equiv u_j - \boldsymbol{u}^T(\boldsymbol{\xi}^{(j)})^*$, $y_{\ell+j} \equiv u_{\ell+j} - \boldsymbol{u}^T(\boldsymbol{\xi}^{(\ell+j)})^*$. Add the inequality

$$-\sum_{i \neq j} (u_i x_i + u_{\ell+i} s_i) - y_j x_j - y_{\ell+j} s_j - u_{2\ell+1}\alpha \leqslant u_0$$

to $\mathcal{F}(C_{k+1})$.
Step 3: Let $k = k + 1$, and go to Step 1.

Note that in iteration k, the algorithm solves $(2\ell|\mathcal{F}(C_k)|)$ LP problems. Also note that the above algorithm makes a very explicit connection between the 'lift-and-project methods' of Balas, Sherali-Adams, Lovász-Schrijver and the notion of 'lifting a valid inequality' as in more traditional polyhedral approaches, see [10]. Below, F denotes the set of feasible solutions of (LCP$_\alpha$), where $\boldsymbol{s} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}\alpha$.

THEOREM 2.2. *Let C_k, $k \in \{1, 2, \dots\}$ be the sequence of convex relaxations generated by Algorithm 2.1. Then $C_\ell = conv(F)$.*

Proof. We think of $\mathcal{K}_k$ for all $k \geqslant 0$, as a subset of $R^{1+(2\ell+1)}$, with the 0th component being the homogenizing variable, the next ℓ components representing $\boldsymbol{x}$, the next ℓ components representing $\boldsymbol{s}$ and the last component representing α. Note that

$$\mathcal{K}_1 \;\subseteq\; \left(\mathcal{K}_0 \cap \left\{\boldsymbol{v} : x_j = 0\right\}\right) + \left(\mathcal{K}_0 \cap \left\{\boldsymbol{v} : s_j = 0\right\}\right)$$

iff

$$\mathcal{K}_1^* \;\supseteq\; \left(\mathcal{K}_0^* + \{-\boldsymbol{e}_j\}\right) \cap \left(\mathcal{K}_0^* + \{-\boldsymbol{e}_{\ell+j}\}\right). \tag{3}$$

(We used the fact that $\mathcal{K}_0 \subseteq R_+^{1+(2\ell+1)}$.) Therefore, if we ensure the latter inclusion, then Theorem 1.4 applies and we can conclude the convergence of the method in ℓ iterations. Recall that every vector $\boldsymbol{u} \in \mathcal{K}_0^*$ represents a valid inequality

$$-\sum_{i=1}^{\ell}(u_i x_i + u_{\ell+i}s_i) - u_{2\ell+1}\alpha \;\leqslant\; u_0$$

for C_0. To ensure the inclusion (3), it suffices to prove:

"For every $\boldsymbol{u}, \boldsymbol{w} \in \mathcal{K}_0^*$ such that
$u_i = w_i,\ \forall i \notin \{j, \ell+j\};\ u_j \geqslant w_j,\ u_{\ell+j} \leqslant w_{\ell+j}$,
we have $\boldsymbol{y} \in \mathcal{K}_1^*$, where $y_i \equiv u_i, \forall i \neq j;\ \ y_j \equiv w_j$."

This is equivalent to proving the fact that if the two inequalities

$$-\sum_{i=1}^{\ell}(u_i x_i + u_{\ell+i}s_i) - u_{2\ell+1}\alpha \;\leqslant\; u_0, \quad \text{and}$$
$$-\sum_{i\neq j}(u_i x_i + u_{\ell+i}s_i) - w_j x_j - w_{\ell+j}s_j - u_{2\ell+1}\alpha \;\leqslant\; u_0$$

are valid for C_0, then

$$-\sum_{i\neq j}(u_i x_i + u_{\ell+i}s_i) - w_j x_j - u_{\ell+j}s_j - u_{2\ell+1}\alpha \;\leqslant\; u_0$$

is valid for C_1. To compute all such inequalities defining C_1, we solve for every valid inequality

$$-\sum_{i=1}^{\ell}(u_i x_i + u_{\ell+i}s_i) - u_{2\ell+1}\alpha \;\leqslant\; u_0$$

for C_0 and every $j \in \{1, 2, \ldots, \ell\}$, the linear programming problems

$$\begin{array}{ll} \text{maximize} & \beta \\ \text{subject to} & \beta\boldsymbol{e}_j + \delta\boldsymbol{e}_{\ell+j} \;\preceq_{\mathcal{K}_0^*}\; \boldsymbol{u}, \end{array}$$

and

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & \kappa \boldsymbol{e}_j + \gamma \boldsymbol{e}_{\ell+j} \preceq_{\mathcal{K}_0^*} \boldsymbol{u}. \end{array}$$

Here, $\preceq_{\mathcal{K}_0^*}$ denotes the partial order induced by the convex cone $\mathcal{K}_0^*$ (that is, $\boldsymbol{u}^1 \preceq_{\mathcal{K}_0^*} \boldsymbol{u}^2$ iff $(\boldsymbol{u}^2 - \boldsymbol{u}^1) \in \mathcal{K}_0^*$). Note that both problems are always feasible. Therefore, each of them either has an optimal solution or is unbounded. If both LPs have optimal solutions, say β^* and γ^* then we set $w_j \equiv u_j - \beta^*$ and $u_{\ell+j} \equiv u_{\ell+j} - \gamma^*$. Since the above two problems are LPs, we can equivalently solve their duals. Namely, we solve the LPs:

$$\begin{array}{lll} (P_j) & \text{minimize} & \boldsymbol{u}^T \boldsymbol{\xi}^{(j)} \\ & \text{subject to} & \xi_j^{(j)} = 1,\ \xi_{\ell+j}^{(j)} = 0,\ \boldsymbol{\xi}^{(j)} \in \mathcal{K}_0, \end{array}$$

and

$$\begin{array}{lll} (P_{\ell+j}) & \text{minimize} & \boldsymbol{u}^T \boldsymbol{\xi}^{(\ell+j)} \\ & \text{subjcet to} & \xi_j^{(\ell+j)} = 0,\ \xi_{\ell+j}^{(\ell+j)} = 1,\ \boldsymbol{\xi}^{(\ell+j)} \in \mathcal{K}_0. \end{array}$$

These latter two linear programming problems are precisely the ones used by Algorithm 2.1. Notice that since their duals are either unbounded or have optimal solutions, these LP problems either have optimal solutions or are infeasible. When (P_j) is infeasible, the equality $x_j = 0$ is valid for F and the algorithm adds this equality to the describing inequalities of C_k. Similarly, when $(P_{\ell+j})$ is infeasible, $s_j = 0$ is valid for F and the algorithm behaves correctly in this instance. (In either instance, the inclusion (3) is obviously satisfied for j.) However, the proof is not yet complete; because, the arguments so far ensure the inclusion (3) when the algorithm is run for every valid inequality of C_0. So, next we prove that what the algorithm does (using only the facets of C_0) suffices. To see this, we need to prove that to derive the facets of $\mathcal{K}_1$, it suffices to start with a facet $\boldsymbol{u}$ of $\mathcal{K}_0$ in the above procedure. Suppose $\boldsymbol{u}, \boldsymbol{w} \in \mathcal{K}_0^*$ satisfy the above conditions but $\boldsymbol{u}$ is not facet inducing for $\mathcal{K}_0^*$. (We will prove that the valid inequality derived from $\boldsymbol{u}$ and $\boldsymbol{w}$ is implied by some other inequalities derived from some facets $\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^\ell$ of $\mathcal{K}_0$.) Since $\boldsymbol{u}$ is not facet inducing for $\mathcal{K}_0$, $\boldsymbol{u}$ is not an extreme ray of $\mathcal{K}_0^*$. Hence, there exist extreme rays $\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^\ell$ of $\mathcal{K}_0^*$ such that for some $\lambda_r > 0$, $r \in \{1, 2, \ldots, \ell\}$, $\sum_{r=1}^{\ell} \lambda_r = 1$ the following conditions are satisfied:

$$\boldsymbol{u} = \sum_{r=1}^{\ell} \lambda_r \boldsymbol{u}^r,\ u_0^r = u_0,\ \forall r \in \{1, 2, \ldots, \ell\}.$$

Note that $\boldsymbol{u}^r$ is facet inducing for each r. Let $\boldsymbol{\xi}^r$ be the optimal solution of (P_j) above for the objective function vector $\boldsymbol{u}^r$. Let $\boldsymbol{\xi}^*$ be an optimal solution of (P_j) when the objective function vector is $\boldsymbol{u}$. We claim that there exists $\tilde{\boldsymbol{\xi}} \in \mathcal{K}_0$ such that

$$(\boldsymbol{u}^r)^T \tilde{\boldsymbol{\xi}} = (\boldsymbol{u}^r)^T \boldsymbol{\xi}^r,\ \forall r \in \{1, 2, \ldots, \ell\},\ \tilde{\xi}_j = 1,\ \tilde{\xi}_{\ell+j} = 0,\ \tilde{\boldsymbol{\xi}} \in \mathcal{K}_0.$$

(This claim follows from Farkas' Lemma, using the facts that $\boldsymbol{u}^r \in \mathcal{K}_0^*, \forall r$ and $\boldsymbol{\xi}^r \in \mathcal{K}_0, \forall r$.) Thus, we have

$$\sum_{r=1}^{\ell} \lambda_r (\boldsymbol{u}^r)^T \boldsymbol{\xi}^r \ = \ \boldsymbol{u}^T \tilde{\boldsymbol{\xi}} \ \geqslant \ \boldsymbol{u}^T \boldsymbol{\xi}^*.$$

Therefore, the inequality obtained from $\boldsymbol{u}$ is equivalent to or dominated by a non-negative combination of the inequalities obtained from $\boldsymbol{u}^r$ which induce facets of $\mathcal{K}_0$. The proof is complete. □

We illustrated a derivation and convergence proof for a successive relaxation method (closely related to Balas' approach and analogous to a suggestion of Lovász and Schrijver [9]) based on Lemma 1.3 and Theorem 1.4. Algorithm 2.1 is an analog of a method based on relaxations $N_0^k(\mathcal{K})$ from [9] (which is concerned with the case of 0-1 integer programming). For the relationship of the methods of [1] and [9], see Balas, Ceria and Cornuejols [2]. (Balas' method [1], in essence, corresponds to defining

$$\mathcal{K}_{k+1} \equiv (\mathcal{K}_k \cap \{\boldsymbol{v} : x_{k+1} = 0\}) + (\mathcal{K}_k \cap \{\boldsymbol{v} : (\boldsymbol{Mx} + \boldsymbol{q}\alpha)_{k+1} = 0\})\ .)$$

Let $C_k^{(4)}$, $k \geqslant 0$ denote the projection of C_k generated by Algorithm 2.1 onto the coordinates

$$\begin{pmatrix} \boldsymbol{x} \\ \alpha \end{pmatrix}.$$

Let $C_k^{(3)}$, $k \geqslant 0$ denote the projection of C_k, generated by Algorithm 1.1, as used in Section 1, onto the coordinates $\begin{pmatrix} \boldsymbol{x} \\ \alpha \end{pmatrix}$. Let $\mathcal{K}_k^{(4)}$ denote the convex cone associated with $C_k^{(4)}$. From the proof of Theorem 2.2, it is easy to see that

$$\mathcal{K}_{k+1}^{(4)} \ = \ \bigcap_{i=1}^{\ell} \left[\left(\mathcal{K}_k^{(4)} \cap \{\boldsymbol{v} : x_i = 0\} \right) + \left(\mathcal{K}_k^{(4)} \cap \{\boldsymbol{v} : s_i = 0\} \right) \right].$$

Therefore, the proofs of Theorems 1.4 and 2.2 imply that

$$\text{if } C_0^{(4)} \supseteq C_0^{(3)} \quad \text{then} \quad C_k^{(4)} \supseteq C_k^{(3)} \ \text{ for all } k \geqslant 0.$$

Thus, the SSILPR method (Algorithm 1.1) as applied in Section 1 to (MIP$_\alpha$) converges at least as fast as Algorithm 2.1 applied to (LCP$_\alpha$).

We have already seen various ways of applying SCRMs to LCP problems. Since the methods proposed in [6, 7] only require a formulation of the feasible solutions by quadratic inequalities, we are also interested in applying the methods of [6, 7] to the following formulation:

$$C_0 \equiv \left\{ \begin{pmatrix} \alpha \\ \boldsymbol{x} \end{pmatrix} \in R^{\ell+1} : \begin{array}{l} \boldsymbol{Mx} + \boldsymbol{q}\alpha \ \geqslant \ \boldsymbol{0}, \ \boldsymbol{x} \ \geqslant \ \boldsymbol{0}, \ \alpha \ \geqslant \ 0, \\ \boldsymbol{e}^T(\boldsymbol{M} + \boldsymbol{I})\boldsymbol{x} + (\boldsymbol{e}^T\boldsymbol{q} + 1)\alpha \ \leqslant \ 1 \end{array} \right\},$$

and

$$\mathcal{P}_F \equiv \{x_i(\boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}\alpha)_i \leqslant 0, \quad i \in \{1, 2, \ldots, \ell\}\}.$$

The general theory of [6] implies that the SSDPR and SSILPR methods converge. It would be interesting to characterize the conditions under which the Algorithms 3.1 and 3.2 of [7] converge in at most ℓ iterations for the above description of $\mathcal{P}_F$ and C_0. Also see [8], where the authors derived some necessary and some sufficient conditions for the finite convergence of SCRMs.

3. A general linear complementarity problem

Let $\mathcal{A} : R^\ell \to R^\ell$, a linear transformation, $\boldsymbol{q} \in R^\ell$ and $\mathcal{K} \subset R^\ell$ a pointed, closed convex cone with nonempty interior, be given. Consider the complementarity problem (CP):

$$\text{(CP) Find } \boldsymbol{x}, \boldsymbol{s} \text{ such that } \mathcal{A}(\boldsymbol{x}) + \boldsymbol{q} = \boldsymbol{s}, \quad \boldsymbol{x} \in \mathcal{K},\ \boldsymbol{s} \in \mathcal{K}^*,\ \langle \boldsymbol{x}, \boldsymbol{s}\rangle = 0,$$

recall that $\mathcal{K}^*$ is the dual of $\mathcal{K}$. Since $\mathcal{K}$ is a pointed, closed convex cone with nonempty interior, so is $\mathcal{K}^*$. Such problems were studied recently, in the context of interior-point methods [14]. We pick $\boldsymbol{\eta} \in \text{int}(\mathcal{K})$, $\bar{\boldsymbol{\eta}} \in \text{int}(\mathcal{K}^*)$ and we can solve instead the optimization problem

$$\begin{array}{lll} (\text{CP}_\alpha) & \text{maximize} & \alpha \\ & \text{subject to} & \boldsymbol{x} \in \mathcal{K},\ \left[\mathcal{A}(\boldsymbol{x}) + \boldsymbol{q}\alpha\right] \in \mathcal{K}^*,\ \alpha \geqslant 0, \\ & & \langle \bar{\boldsymbol{\eta}}, \boldsymbol{x}\rangle + \langle \boldsymbol{\eta}, \mathcal{A}(\boldsymbol{x}) + \boldsymbol{q}\alpha\rangle + \alpha \leqslant 1, \\ & & \langle \boldsymbol{x}, \mathcal{A}(\boldsymbol{x}) + \alpha\boldsymbol{q}\rangle = 0. \end{array}$$

We choose

$$C_0 \equiv \left\{ \begin{pmatrix} \alpha \\ \boldsymbol{x} \end{pmatrix} \in R^{\ell+1} : \begin{array}{l} \boldsymbol{x} \in \mathcal{K},\ \left[\mathcal{A}(\boldsymbol{x}) + \boldsymbol{q}\alpha\right] \in \mathcal{K}^*,\ \alpha \geqslant 0, \\ \langle \bar{\boldsymbol{\eta}}, \boldsymbol{x}\rangle + \langle \boldsymbol{\eta}, \mathcal{A}(\boldsymbol{x}) + \boldsymbol{q}\alpha\rangle + \alpha \leqslant 1 \end{array} \right\}.$$

Note that C_0 is always a compact convex set (see the next theorem). We also pick

$$\mathcal{P}_F \equiv \left\{ \langle \boldsymbol{x}, \mathcal{A}(\boldsymbol{x}) + \alpha\boldsymbol{q}\rangle,\ -\langle \boldsymbol{x}, \mathcal{A}(\boldsymbol{x}) + \alpha\boldsymbol{q}\rangle \right\}.$$

THEOREM 3.1.

(i) *C_0 is a compact convex set.*

(ii) *(CP$_\alpha$) has an optimal solution with $\alpha^* > 0$ iff (CP) has a solution (or solutions).*

(iii) *If $\begin{pmatrix} \alpha^* \\ \boldsymbol{x}^* \end{pmatrix}$ is an optimal solution of (CP$_\alpha$) with $\alpha^* > 0$ then the pair of vectors $\left(\dfrac{\boldsymbol{x}^*}{\alpha^*}, \dfrac{1}{\alpha^*}\mathcal{A}(\boldsymbol{x}^*) + \boldsymbol{q}\right)$ solves (CP).*

Proof.

(i) We only need to show that C_0 is bounded; because, C_0 is a closed and convex subset of $R^{\ell+1}$ by definition. Assume on the contrary that we can take an unbounded direction $\begin{pmatrix}\Delta\alpha\\ \Delta\boldsymbol{x}\end{pmatrix} \neq \mathbf{0}$ in C_0;

$$\begin{aligned}&\begin{pmatrix}\Delta\alpha\\ \Delta\boldsymbol{x}\end{pmatrix} \neq \mathbf{0},\ \ \Delta\boldsymbol{x} \in \mathcal{K},\ \Delta\alpha \geqslant 0,\ \left[\mathcal{A}(\Delta\boldsymbol{x}) + \boldsymbol{q}\Delta\alpha\right] \in \mathcal{K}^*,\\ &\langle\bar{\boldsymbol{\eta}}, \Delta\boldsymbol{x}\rangle + \langle\boldsymbol{\eta}, \mathcal{A}(\Delta\boldsymbol{x}) + \boldsymbol{q}\Delta\alpha\rangle + \Delta\alpha \ \leqslant\ 0.\end{aligned}$$

Since each term in the left hand side of the last inequality is nonnegative, we have

$$\langle\bar{\boldsymbol{\eta}}, \Delta\boldsymbol{x}\rangle = 0 \ \text{ and } \Delta\alpha = 0.$$

Since $\bar{\boldsymbol{\eta}} \in \text{int}(\mathcal{K}^*)$ and $\Delta\boldsymbol{x} \in \mathcal{K}$, the first identity above implies that $\Delta\boldsymbol{x} = \mathbf{0}$. Thus, we have a contradiction to $\begin{pmatrix}\Delta\alpha\\ \Delta\boldsymbol{x}\end{pmatrix} \neq \mathbf{0}$.

(ii) Suppose (CP$_\alpha$) has an optimal solution $\begin{pmatrix}\alpha^*\\ \boldsymbol{x}^*\end{pmatrix}$ with $\alpha^* > 0$. Then $\bar{\boldsymbol{x}} \equiv x^*/\alpha^* \in \mathcal{K}$, $\bar{\boldsymbol{s}} \equiv (1/\alpha^*)\mathcal{A}(\boldsymbol{x}^*) + \boldsymbol{q} \in \mathcal{K}^*$. We have

$$\langle\bar{\boldsymbol{x}}, \bar{\boldsymbol{s}}\rangle \ = \ \langle\bar{\boldsymbol{x}}, \mathcal{A}(\bar{\boldsymbol{x}}) + \boldsymbol{q}\rangle \ = \ \frac{1}{(\alpha^*)^2}\langle\boldsymbol{x}^*, \mathcal{A}(\boldsymbol{x}^*) + \alpha^*\boldsymbol{q}\rangle = 0.$$

Therefore, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{s}})$ solves (CP). For the converse, let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{s}})$ be a solution of (CP). Let

$$\zeta \equiv \langle\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{x}}\rangle + \langle\boldsymbol{\eta}, \bar{\boldsymbol{s}}\rangle \ \geqslant\ 0,\ \alpha^* = \frac{1}{\zeta + 1} \ \text{ and } \boldsymbol{x}^* = \frac{\bar{\boldsymbol{x}}}{\zeta + 1}.$$

Then $\begin{pmatrix}\alpha^*\\ \boldsymbol{x}^*\end{pmatrix}$ is a feasible solution of (CP$_\alpha$). But the feasible region of (CP$_\alpha$) is compact and nonempty, its objective function is linear, hence, (CP$_\alpha$) has optimal solution (or solutions). Since we already showed a solution with positive objective value, the optimum value is positive.

(iii) This claim follows from the proof of (ii).

□

Theorem 3.1 shows that we can apply SCRMs to (CP$_\alpha$) with the above C_0 and $\mathcal{P}_F$ and solve the original, general problem (CP).

4. Conclusion

We reviewed and presented various ways of using SCRMs to solve LCPs. Our point of view was theoretical. In Section 2, we first showed how to extend Pardalos-Rosen formulation to a more compact formulation of LCP. Then, we applied the ideas of Balas, Sherali-Adams and Lovász-Schrijver to derive a similar method with the same nice property of termination within ℓ major iterations. Our approach does not make any assumptions on the data of the LCP. However, this is only a theoretical improvement over the existing work of [13]. Some of the insights, the algorithm of Section 2, the formulations of Sections 2 and 3 may help improve some practical applications; but, a successful computational procedure will still have to incorporate techniques like branch-and-bound and/or those of [13]. In Section 3 we provided yet a more theoretical new result and proved (by very elementary means) that SCRMs and (more importantly) the convergence theory of [6, 7] are applicable to a very general class of complementarity problems.

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